Self-Tuned Regenerative Amplification and the Hopf Bifurcation

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Abstract—Recent work in cochlear amplifier modeling has focused on systems which show the dynamics of a Hopf bifurcation. We show that these systems are examples of a generic amplifier topology, the self-tuned regenerative amplifier (STRA). The STRA is a feedback-stabilized regenerative amplifier that can be operated in a region of supercritical stability. The signatures of Hopf amplification, such as a cubic nonlinear small-signal response at resonance, are general features of the topology. The topology is shown to include a degenerate parametric amplifier, which may explain its low noise and insensitivity to input-feedback phase mismatch.

I. INTRODUCTION

One of the ongoing mysteries of mammalian hearing is the way in which active amplification takes place in the cochlea. The existence of electromechanical transduction was hypothesized by Gold in 1948 [1] and proven in 1978 by the detection of cochlea-generated sounds – so-called otoacoustic emissions [2]. During the last decade, it has been established that the sources of this electromechanical energy are the outer hair cells (OHCs) [3]. There remain a number of puzzles regarding the biophysics of this amplification, not least being how the OHCs are able to phase-lock to the vanishingly small input signals which they amplify (the sensitivity of human hearing is such that successful detection occurs at the level of thermal noise in the system [4]); and why these systems show a cubic nonlinear response even at vanishingly small signal amplitudes, when operating at their resonant frequency.

Recent models suggest that the cochlear amplifier (CA) operates dynamically on or near to a Hopf bifurcation [5-7], which is considered to explain both its sensitivity and its nonlinearity (logarithmic compression in amplitude response. These models are complicated by the necessity to represent a continuous multidimensional biological structure using a lumped-component dynamical system, and the effect of coupling between lumped-component resonators has been shown to significantly affect the results. In a companion paper to this [8] we show that in a real aVLSI-implemented cochlea, as opposed to a theoretical model, common simplifying assumptions (such as modeling the vestibular fluid as propagating waves in one direction only) are not supportable.

The Hopf bifurcation dynamics are not unique to cochlear amplifiers and also appear in other electromechanical systems [9, 10]. In this paper we present a generic topology for all these systems, the self-tuned regenerative amplifier (STRA). We show that when this amplifier is operating in the supercritical stability region (close to the Hopf bifurcation) it shows the signature features which have previously been attributed to unique physical or electrochemical structures in the disparate implementations. In particular, we focus on the gain close to the bifurcation itself, and pay particular attention to the effect of frequency on the small-signal response. We also show that the STRA has significant similarity to the class of degenerate parametric amplifiers, and that the advantages of these amplifiers, such as low noise and robustness to input-feedback phase mismatch, carry over to the STRA.

The STRA is an extension of the traditional regenerative [11] and superregenerative topologies [12], and has also been described as a supercritically stable regenerative amplifier (SSRA) [13]. Significantly, Gold (who was a physicist, rather than a physiologist) originated the CA hypothesis after several years of working on regenerative amplifiers, and this topology has always seemed to offer a good explanation of the CA’s probable microelectromechanical operation.

II. THE SELF-TUNED REGENERATIVE AMPLIFIER

The basic topology of the STRA is shown in Figure 1. It can be described as a positive feedback amplifier which continually adjusts its gain to remain stable, and to maximally amplify small signals. These aims – stability and maximum amplification – are contradictory, so the amplifier self-tunes to the boundary of stability (the point of the Hopf bifurcation).

The STRA consists of a frequency selective (bandpass) structure which is normally modeled as a second-order resonant system. This system is driven by a wideband linear amplifier (WLA) with output gain which is adjustable by means of a multiplier. The magnitude of the signal of interest is measured in the resonant system, and this magnitude is fed
back and used to adjust the gain of the WLA. At the same time, the signal of interest is fed directly back to the input of the WLA, and is amplified and used to drive the resonant structure. Given that this feedback is positive, the system will self-oscillate unless the gain is appropriately controlled. In practice, the operating point is on the cusp of instability, hence the use of the term supercritical stability – the conventional description of stability on a Hopf bifurcation [14].

The topology shown in Fig. 1 may be varied without changing the principle of operation; for example, Fig. 1 shows the signal magnitude represented by a signal squaring block, but in practice [13] an RMS-DC converter could be used.

The amplifier operates as follows: assume that we wish to measure the response of a resonant system to a forcing function \( F \) in terms of the variable of interest, \( x \), so that our system would be described by:

\[
F + Gx = M \frac{d^2 x}{dt^2} + \zeta \frac{dx}{dt} + kx
\]

where \( G \) is the gain of the regenerative path, and \( M, \zeta \) and \( k \) are parameters of the resonant system corresponding to energy storage, dissipation and restoring force respectively. Given that \( G \) is defined by:

\[
G = g(\mu - |x|^2)
\]

where \( g \) is the gain of the WLA and \( \mu \) is the system setpoint, we can rewrite (1) as:

\[
M \frac{d^2 x}{dt^2} + \zeta \frac{dx}{dt} + (k - g\mu)x + g|x|^2 = F = 0.
\]  

This is the classic form of a dynamical system displaying a Hopf bifurcation [5-7, 14]. In particular, the presence of the term in \( |x|^2 \) guarantees a small-signal cubic response at the resonance frequency, which is considered to be the signature of Hopf dynamics in cochlear amplification. Supercritical stability would occur (theoretically) at the operating point where \( k = g\mu \), and the linear term in \( x \) would drop out completely, leaving only the cubic term.

There are two alternative forms of (3) corresponding to the cases where the amplified variable is the first or second derivative of \( x \), as follows:

\[
M \frac{d^2 x}{dt^2} + \zeta \frac{dx}{dt} + g(\mu - |x|^2) = F = 0
\]

(4)

\[
M \frac{d^2 x}{dt^2} + \zeta \frac{dx}{dt} + g(\mu - |x|^2) = F = 0.
\]  

These three variations implement amplification with potential instability created by negative restoring force, negative energy storage or negative damping respectively. We will show by means of a physical example how these variations may be applied. At this stage, we wish to draw the reader’s attention to the fact that the cubic term is inherent in the topology, rather than a function of the physical implementation. It might be thought that the \( |x|^2 \) block is a contrived way in which to achieve this, but any system element which squares or simply rectifies the signal will be adequate; and if this element low-pass filters the squared signal (as in an RMS-DC converter) that is also acceptable.

III. EXAMPLE: A SERIES RLC SYSTEM

Many transducers and antennae detect their signals either by filtering via an RLC network, or by converting the signal to a variation in resistivity, capacitance or inductance. If our measurand modulates the current, and we measure the current in the usual way by sensing the potential difference across the resistor \( R \), our system takes the form of (4) above:

\[
L \frac{d^2 i}{dt^2} + (R - g\mu) \frac{di}{dt} + g \frac{i}{di} \frac{di}{dt} + \frac{i}{C} = F = 0.
\]  

There are two significant points at which this system can operate – at supercritical stability \( (R = g\mu) \), and with the input at the resonant frequency \( (\omega_0 = 1/(\sqrt{LC})) \). At resonance the inductive and capacitive impedances cancel each other, leaving:

\[
(R - g\mu) \frac{di}{dt} + g \frac{di}{dt} \frac{di}{dt} = F = 0.
\]  

At supercritical stability the first term in (7) drops out too, leaving only the cubic term. We can solve (6) for sinusoidal input – at supercritical stability and resonance it is:

\[
i(t) = \frac{4F(t)}{3g\mu_0^2}.
\]

This theoretical supercritically stable point is that at which the circuit damping is exactly cancelled or balanced by the input energy (“negative damping”), in the absence of noise. In
practice, noise would push the system into oscillation, and the feedback circuit would automatically reduce the gain to damp the noise-induced oscillations. Nonetheless, we see that in the limit of small-signal operation at resonance, the circuit response is a cubic rather than linear function of the input forcing.

![Figure 2](image)

Figure 2. The response of the STRA (6) at resonance for different values of \( \mu \) in equally spaced steps of \( \mu g = 0.1R \) from \( \mu g = 0 \) (no feedback – lowest curve) to \( \mu g = R \) (supercritical stability – uppermost curve). The progression from the linear response to the cubic response is very clear. Circuit parameters were \( g=10 \), \( R=\Omega \), \( L=500 mH \), \( C=50 nF \); simulation performed in MATLAB.

![Figure 3](image)

Figure 3. The response of the STRA (6) for a range of frequencies on either side of resonance (6324 Hz), and for inputs ranging from 100 \( \mu V \) (lowest curve) to 6 mV (uppermost curve), for \( \mu g = 0.9R \). It can be seen that the effective Q of the system diminishes towards its open-loop value as the input amplitude increases. The heavier line at top is the open loop response for the highest input level – it can be seen that there is negligible difference between open and closed loop systems at that level. Circuit parameters were \( g=10 \), \( R=\Omega \), \( L=500 mH \), \( C=50 nF \); simulation performed in MATLAB.

Figure 2 shows the response of the system in (6) at resonance for different values of the control parameter or system setpoint \( \mu \). It can be seen that as we reduce the control parameter, the response becomes less cubic and approaches linearity.

Figure 3 shows the response for the system across a range of frequencies near resonance, and at various levels of input signal amplitude. The feedback setpoint is \( \mu g = 0.9R \), such as might reasonably be used in practice. The effect of the positive feedback can be seen in the increase in Q as the input signal is reduced. At high input levels, the feedback is essentially minimal, and the Q of the closed-loop system is very similar to the Q of the basic resonator.

A. The STRA as a parametric amplifier

We can think of the STRA’s combination of a linear amplifier and gain-adjusting multiplier as forming a degenerate parametric amplifier, as shown in Fig. 4. The squared signal (e.g. \( \sin^2 \omega t \)) can be thought of as a signal with a DC bias and a quadrature phase-shifted sinusoidal component at twice the input frequency \((1+\cos 2\omega t)/2\), hence creating an ideal pump frequency for degenerate parametric amplification.

![Figure 4](image)

Figure 4. The wideband linear amplifier WLA and multiplier form a parametric amplifier, with the gain adjust signal forming the pump input to the amplifier.

The recasting of this amplifier as a type of parametric amplifier has two significant consequences: it goes some way to explaining the low noise performance of the STRA system (which is unexpected, given that noise might be expected to propagate from the resonant circuit to the input of the WLA), and it suggests that the system will be robust to a wide range of phase shift between input and pump signals, as shown in Fig. 5. It has been suggested [15] that the cochlear amplifier may be a parametric amplifier, although the view has not gained widespread support. Here we see that the standard Hopf hypothesis suggests inherently the presence of parametric amplification.

Working from Eq. (2) we can split the overall gain into two components, a linear gain and a parametric gain, as shown below for \( x(t) = X \sin \omega t \):

\[
G = g\left[\mu - X^2 \sin^2 \omega x\right]
\]

(9)

where the first term in parentheses is the linear gain and the second term is the parametric gain (the term \( \text{linear} \) is used here in the sense of amplifier classification – the output proportional to the input - rather than mathematics; it is still dependent on \( X^2 \)). The linear gain will be independent of the phase or frequency of the amplifier input signal, but the parametric gain will be phase and frequency dependent.

There are two potential phase shifts to consider: phase shifts introduced in the signal paths (such as in the squaring element of Fig. 1) and phase shift between an existing signal in the feedback loop and an incoming signal. In the event that there is a phase shift between the squared (pump) signal and the feedback (input) signal, over and above the \( \pi/2 \) implied by the basic trigonometry, the gain of the parametric amplifier...
will be reduced. In the following section we show that a typical phase-shift-inducing element, such as the low pass filter in an RMS-DC converter, may have such an effect; but that there remains a wide range of phase for which amplification will take place.

B. Use of an RMS-DC converter

In real biological or physical systems there is unlikely to be a pure squaring function available as depicted in Fig. 1. However, there are many processes which approximate this; for example, in the human cochlea, inner hair cells (IHCs), which would act as the feedback sensing block, generally have a nonlinear monopolar response to bipolar deflection, approximating to rectification; and at higher frequencies (above 1kHz) their output has a significant DC component (the summation potential). Above 5kHz this DC component of response dominates the output and the AC phase-locked component is regarded as negligible.

An RMS-DC converter with input $v_{rms}(t)$ and output $v_{DC}(t)$ with an averaging (leaky low pass) filter defined by the transfer function

$$H(s) = \frac{s\tau_1 + 1}{\tau_1\tau_2}$$

will have a response of the form:

$$\frac{dv_{DC}^2(t)}{dt} = -\frac{v_{DC}^2(t)}{\tau_1} + \frac{v_{rms}^2(t)}{\tau_2}.$$ (11)

For $v_{rms}(t) = Re^{j\omega t}$ we will have $v_{DC}(t) = A + Be^{j2\omega t + \phi}$ where $A$, $B$ and $\phi$ can be solved in the usual way. We are most interested in the phase of the oscillatory component of $v_{DC}(t)$:

$$\phi = \arctan\left(\frac{2\omega}{\tau_1}\right)$$ (12)

which will vary between 0 and $\pi/2$ as might be expected. We can adapt a method from Rugar and Grütter [9] to calculate the parametric gain with respect to phase shift.

For the system in Eq. (3) the relationship between the parametric gain and the phase difference (between input and pump signals) is:

$$G_p(\phi) = \left[\frac{\cos^2 \phi + \sin^2 \phi}{1 + H'^2}\right]^{1/2}$$ (13)

where

$$H = -m\omega_0 gX^2.$$ (14)

Fig. 5 shows the gain of an amplifier such as that in Fig. 1, for a full range of possible phase shifts.

IV. CONCLUSIONS

The STRA amplifier is a versatile nonlinear amplifier that includes a parametric amplification component, and creates systems with the dynamical signature of a Hopf bifurcation. A wide range of physiological and physical systems may include structures of this type, and the presence of Hopf bifurcation dynamics may indicate the presence of such a structure.

REFERENCES