General theory of carrier lifetime in semiconductors with multiple localized states

Didier Debuf

Center of Excellence for Advanced Silicon Photovoltaics and Photonics, The University of New South Wales, Sydney, New South Wales 2052, Australia

(Received 28 June 2004; accepted 16 August 2004; publisher error corrected 3 January 2005)

The Shockley-Read-Hall rate equations determine the average carrier transitions via a single-level defect in the band gap of a nondegenerate semiconductor. In the present work the differential rate equations for multiple levels, or localized states systems, are derived from first principles. These multiple level systems comprise the multiple discrete defects system and the coupled or excited states system. The solution for the single-level rate equations, developed recently for transient decay, is represented by an infinite series of monoexponential terms, the frequencies or inverse time constants of which are a linear combination of the fundamental frequencies \(\omega = 1/\tau\). For the multiple localized state solution expressions for the fundamental time constants \(\tau_{1+1}\) are derived for \(m\) states with \(k = 1, 2, \ldots, m\) without an approximation at a given temperature for an excess carrier concentration below nondegenerate doping, arbitrary uniform doping concentration \(N_{A,D}\), defect level concentration \(N_k\), cross section \(\sigma_{sk,pk}\), and energy level \(E_k\). Verification of the set of rate equations for each system is performed by comparing the analysis of the numerical solution for component time constants with the prediction of the theory. The variation of the fundamental time constant \(\tau_1\) with excess carrier concentration indicates the behavior of minority carrier trapping. © 2004 American Institute of Physics. [DOI: 10.1063/1.1805192]

I. INTRODUCTION

The Shockley-Read-Hall (SRH) rate equations of Refs. 1 and 2 describe the evolution with time of recombination, capture, and emission of excess carriers via a single defect energy level in the band gap of a semiconductor. These thermal and nonradiative processes determine the average lifetime of minority carriers. For compound semiconductors, as with silicon, SRH recombination is a dominant loss mechanism and for low band gap semiconductor materials, Auger recombination may dominate over SRH recombination. For the single-level defect which may be considered as a localized state in \(K-p\) space, the near equilibrium solution of Ref. 3 is extended to all excess carrier concentrations below nondegenerate doping by the recent exact solution of Ref. 4. The present work addresses the derivation of the differential rate equations from first principles, for multiple localized states without approximations for excess carrier concentrations away from the equilibrium point. The method of solution developed in Ref. 4 for the single-level, is applied to the multiple localized states case, where there are \(m\) states. The validity of these sets of rate equations is confirmed by first, predicting time constants from the solution, and second analyzing the numerical solution for component decay time constants using a multitransient technique.

Consider the decay response of a semiconductor to a light impulse \(N_0 \delta(t)\). The excess carriers, generated by the light, decay with time constants due to the energy level depth \(E_k\), concentration \(N_k\), and cross sections \(\sigma_{n,p}\) of the \(k\) multiple localized states in the band gap. Two localized state systems that affect the decay time constants are the multiple discrete system and the multiple excited state system. The single quantum well (SQW) and the multiquantum well (MQW) structures may be considered within the framework of localized states. Carrier transitions between states in the multiple discrete system are not probable. For the multiple excited states or coupled system the carrier transitions between the states may be selective. Evidence for this is found in Refs. 5 and 6 for the phosphorus donor and a preferred path to a vacancy-oxygen complex or carbon interstitial oxygen interstitial in Czochralski-grown silicon. Also, hopping effects between defect levels in gallium nitride have also been proposed. Furthermore, in SQW and MQW structures the carrier transitions follow selection rules dependent on the direction of the electric field \(\xi\).

The multiple localized state analytic solution presented in this work indicates that there are an infinity of monoexponential terms present in the decay, the frequencies (inverse time constants) of which are a linear combination of \(m+1\) fundamental frequencies \(\omega_{1+k} = 1/\tau_{1+k}\). As with the single-level solution of Ref. 4 the fundamental frequencies or inverse time constants are derived from the linear form of the scaled rate equations. Expressions for the minority \((\tau_1)\) and majority \((\tau_{1+k}\) with \(k = 1, 2, \ldots, m\) carrier decay time constants, are given. The fundamental time constants \(\tau_{1+k} = 1/\omega_{1+k}\) form a set decreasing in magnitude with \(\tau_1 > \tau_2 > \tau_3, \ldots, \tau_{1+k}\). Fundamental frequencies \(\omega_{1+k}\) representing \(m\) energy levels, are derived for isothermal conditions for arbitrary excess carrier concentration below nondegenerate doping. Uniform doping concentration \(N_{A,D}\), defect level concentration \(N_k\), cross section \(\sigma_{sk,pk}\), and energy level \(E_k\). The theory presented here addresses the whole decay process from \(0^+ < t < \infty\). Since each monoexponential term is the re-
In Sec. II. These differential equations are derived from first principles for the discrete state systems in Appendix A and the coupled or excited states system in Appendix B. It is shown that the nonlinear terms in the rate equations for scaled or normalized excess carrier concentration $\Delta n(t)$ and $\Delta p(t)$ go to zero at some time $t=t_0$. The resulting differential equations may be solved at $t=t_0$ for the fundamental frequencies or eigenvalues $-\omega_{1+k}$.

The fundamental frequencies are evaluated from the determinant of the matrix of linear differential equations for the multiple localized state systems. A discrete state level system is evaluated in Sec. III and the excited state level system is evaluated in Sec. IV. Having obtained the fundamental time constants, a general analytical solution for $\Delta n(t)$ and $\Delta p(t)$ representing the impulse response is given.

Section V contains a multitransient analysis of the numerical solution, for both the discrete state system and the coupled state system, to determine the component exponentials in the sum of exponentials. Multitransient analysis (the derivation of component exponentials in a sum of exponentials) of the numerical solution yields a good agreement with the analytical solution indicated above. In particular the linear combination of fundamental frequencies in the exponents predicted by the analytic solution and the variation of the time constant $\tau_1=1/\omega_1$ with excess carrier concentration, indicates close agreement. For this variation of $\tau_1$ recombination in the linear region and trapping in the nonlinear region are discussed.

Experimental evidence for the theory presented is given in Secs. VI and VII. The constraints on realizing the fundamental frequencies from an experimental situation for the discrete state and excited state systems are discussed in Sec. VI. In particular, a low concentration of excited state levels may yield a long fundamental time constant $\tau_1$. The semiconductor silicon sample properties and experiment are discussed in Sec. VII.

In Sec. VIII a discussion emphasizes the implications of the theoretical results determined above. In particular, defect level parameters for the same level as in Ref. 4 are retained in the present analysis and it is shown that with additional levels the dominant time constant $\tau_1$ is lower. This is to be expected, since the decay constant $\tau_1$ represents the minority carrier decay. The conclusions in Sec. IX indicate the uniqueness of the fundamental frequencies for a given set of localized state parameters.

II. GENERAL EQUATIONS: DERIVATION OF FREQUENCIES $\omega$

The two localized states or defect types considered are the multiple discrete states system and the coupled or ground excited states system. These two defect systems are considered to have the major effect on excess carrier recombination and it is for this reason that a method of predicting their effect on the decay time constants is presented. The rate equations describe the average energy position of the states in the band gap. These nonlinear differential equations may be solved, using the method of Ref. 4, for the inverse decay time constants or fundamental frequencies. It is shown in this
section that the eigenvalues or fundamental frequencies which correspond to multiple states, may be evaluated at time \( t = t_0 \) where the nonlinear terms vanish, in the nonlinear differential rate equations.

With reference to Fig. 1, consider \( m \) localized states with energy state \( E_k, k = 1, 2, \ldots, m \), in the band gap of a semiconductor. For this case there is no interaction between states and the localized states may be regarded as discrete. Figures 1 and 2 together show states with energy \( E_k, k = 1, 2, \ldots, m \), representing the “ground state” and states \( E_s \), representing “excited states,” in the band gap of a semiconductor indicating an interaction between states. This system represents the coupled or excited states system.

A first principles analysis applied to the derivation of the carrier transitions, indicated in Fig. 1, is given in Appendix A. This derivation is in agreement with the rate equations (1)–(3) given in Refs. 7 and 8 for the discrete states system. A similar first principles analysis for the coupled states system is presented in Appendix B. The following Eqs. (1)–(3) apply for the recombination via and emission from the respective states indicated in Figs. 1 and 2 with \( n(t) \) and \( p(t) \) the electron and hole concentrations in the respective band, and \( f_k(t) \) the \( k \)th level occupancy function:

\[
\frac{dn(t)}{dt} = \sum_{k=1}^{m} \left[ e_{nck}n_kf_k(t) - C_{nck}n(t)N_k[1 - f_k(t)] \right], \tag{1}
\]

\[
dp(t) = \sum_{k=1}^{m} \left[ e_{pck}n_kf_k(t) - C_{pck}p(t)N_k[1 - f_k(t)] \right], \tag{2}
\]

\[
N_k \frac{df_k(t)}{dt} = \sum_{k=1}^{m} \left[ C_{nck}n(t) + e_{pck}N_k \right] - \left[ C_{nck}n(t) + e_{pck}n_k + e_{nck}N_kf_k(t) \right] + \sum_{k'=1,k'+k} (U_{nk'k} - U_{nkk'}) (U_{pk'k} - U_{pkk'}). \tag{3}
\]

In terms of the notation of Fig. 2, with regard to Eq. (3) for the coupled states system, see Appendix B Eq. (B15) for the relevant expressions for the summation of \( U_{k'k}^{np} = (U_{nk'k} - U_{nkk'}) \) and \( U_{k'k}^{np} = (U_{nkk'} - U_{pk'k}) \).

In Eqs. (1)–(3), the subscript \( n \) or \( p \) denotes the carrier type so that \( C_{nck} = v_{thk} \sigma_{nck} \) and \( C_{pck} = v_{thp} \sigma_{pck} \) represent the capture coefficients for electrons and holes respectively. The subscripts \( c \) and \( v \) represent the conduction and valence bands and \( k \) represents the \( k \)th state in the band gap. The direction of the carrier transition is described by \( ck \) or \( vk \) being the conduction band to the \( k \) state or the valence band to the \( k \) state, respectively. The quantities \( \sigma_{nck} \) and \( \sigma_{pck} \) represent the cross sections for electrons and holes from the conduction and valence bands to the quantum state, respectively. The quantities \( v_{thn} \) and \( v_{thp} \) are the average thermal velocity of electrons and holes, respectively. In terms of the notation of Fig. 1, for example, the capture rate is \( e_{nck} = C_{nck}N_k \), the emission rate is \( e_{nck} = C_{nck}N_k \), and \( N_k \) is the concentration of the level one state. The electron and hole emission rates are defined by \( e_{nck} = C_{nck}P_{1k} \) and \( e_{pck} = C_{nck}P_{1k} \), respectively, where \( n_{1k} \) and \( p_{1k} \) are given by:

\[
n_{1k} = N_e \exp \left( -\frac{(E_e - E_k)}{kT} \right), \tag{4a}
\]

\[
p_{1k} = N_e \exp \left( -\frac{(E_e - E_k)}{kT} \right). \tag{4b}
\]

The differential rate equations (1) and (2) are scaled (see Ref. 9, for example) by the normalizing factor \( \Delta n(0) = \Delta p(0) \) resulting in normalized terms \( \Delta n_{nm}(t) \) and \( \Delta p_{nm}(t) \), where the subscript \( nm \) indicates a normalized quantity representing the scaled electron and hole concentrations. The charge neutrality relation is expressed in normalized terms as

\[
\Delta p_{nm}(t) = \Delta n_{nm}(t) + \sum_{k=1}^{m} \frac{N_k}{\Delta n(0)} \Delta f_k(t). \tag{5}
\]

The scaled differential equations (1) and (2) are given by Eqs. (6) and (7) respectively, and written in a similar form to that of Ref. 10.
\[- \frac{d\Delta n_{nm}(t)}{dt} = \sum_{k=1}^{m+4} \left[ \Delta n_{nm}(t)(\gamma_{ok} - \sigma_{ok}) - \frac{\sigma_{ok}}{\Delta n(0)} N_k \Delta f_k(t) \right. \]
\[\left. - C_{nck}\Delta n(0)N_k\Delta f_k(t)\Delta n_{nm}(t) \right], \tag{6}\]
\[- \frac{d\Delta p_{nm}(t)}{dt} = \sum_{k=1}^{m+4} \left[ \Delta p_{nm}(t)(\alpha_{ok} - \beta_{ok}) + \frac{\beta_{ok}}{\Delta n(0)} N_k \Delta f_k(t) \right. \]
\[\left. + C_{pcnk}\Delta n(0)N_k\Delta f_k(t)\Delta p_{nm}(t) \right], \tag{7}\]

where the symbols \(\alpha_{ok}, \beta_{ok}, \gamma_{ok}, \) and \(\sigma_{ok}\) are given as follows:
\[
\alpha_{ok} = C_{pcnk}N_k \left[ \frac{p_{1k}}{p_{po} + p_{1k}} + \frac{p_{po} + p_{1k}}{N_k} \right], \tag{8a}\]
\[
\beta_{ok} = C_{pcnk}N_k \left[ \frac{p_{po} + p_{1k}}{N_k} \right], \tag{8b}\]
\[
\gamma_{ok} = C_{ncnk}N_k \left[ \frac{n_{1k}}{n_{po} + n_{1k}} + \frac{n_{po} + n_{1k}}{N_k} \right], \tag{8c}\]
\[
\sigma_{ok} = C_{ncnk}N_k \left[ \frac{n_{po} + n_{1k}}{N_k} \right]. \tag{8d}\]

It is shown that the nonlinear terms in the rate equations (6) and (7) for \(\Delta n_{nm}(t)\) and \(\Delta p_{nm}(t)\), respectively, go to zero at some time \(t=t_0\). The resulting equations contain terms in only the eigenvalues or fundamental frequencies \(\omega_{1+k}\). As there are \(m+1\) equations and \(m+1\) unknowns the equations may be solved for the fundamental frequencies \(\omega_{1+k}\). In order to determine the eigenvalues for all excess carrier concentrations, the behavior of the defect occupancy function \(\Delta f_k(t)\) for all \(k\), relative to \(\Delta n_{nm}(t)\), is first examined with respect to the three discrete defect levels or quantum states system. For this example, the terms \((U_{nk}−U_{nkk})\) and \((U_{pk}−U_{pkk})\) are set to zero in Eq. (3) as the states are discrete. This is extended to the coupled states case in Sec. IV where there are carrier transitions between the quantum states and these terms are retained in the rate equations.

It is shown that the nonlinear terms in the second order differential equation describing \(\Delta n_{nm}(t)\) vanish at time \(t=t_0\) for two conditions. This allows the eigenvalues to be determined. For a given excess carrier concentration \(\Delta n(0)\), one defect level occupancy function \(\Delta f_k(t)\) for some \(k\) may intersect \(\Delta n_{nm}(t)\) at some time \(t_0\). A further increase in \(\Delta n(0)\) may result in two defect level occupancy functions intersecting \(\Delta n_{nm}(t)\) concurrently at \(t=t_0\). These two intersections occur simultaneously at \(t=t_0\), although the times \(t_0\) for a given \(k\) of each intersection are different.

Two variables are defined and the conditions on these two variables determine how the nonlinear terms in the second order differential equation go to zero at time \(t=t_0\). Let \(\eta_{nk}(t_{i_k})=\Delta f_k(t_{i_k})=\Delta n_{nm}(t_{i_k})\) at the point of intersection for some \(k\). There is also a corresponding term \(\eta_{pk}(t_{i_k})\) to be described below. Note that the occupancy functions intersecting simultaneously with \(\Delta n_{nm}(t)\) at \(t=t_0\), the values of \(\eta_{nk}(t_{i_k})\) and \(\eta_{pk}(t_{i_k})\) are from intersections that occur at different \(t_{i_k}\). Equations (6) and (7) may be written as Eqs. (9) and (10), respectively, which include the terms \(\eta_{nk}(t_{i_k})\) and \(\eta_{pk}(t_{i_k})\).

\[- \frac{d\Delta n_{nm}(t)}{dt} = \sum_{k=1}^{m} \left[ \Delta n_{nm}(t)(\gamma_k - \sigma_k) - \frac{\sigma_k}{\Delta n(0)} N_k \Delta f_k(t) \right. \]
\[\left. - C_{nck}\Delta n(0)N_k\Delta f_k(t)[\Delta n_{nm}(t) - \eta_{nk}(t)] \right]. \tag{9}\]

Similarly for Eq. (7) substitution of a corresponding set of terms containing \(\eta_{pk}(t_{i_k})\) results in

\[- \frac{d\Delta p_{nm}(t)}{dt} = \sum_{k=1}^{m} \left[ \Delta p_{nm}(t)(\alpha_k - \beta_k) + \frac{\beta_k}{\Delta n(0)} N_k \Delta f_k(t) \right. \]
\[\left. + C_{pcnk}\Delta n(0)N_k\Delta f_k(t)[\Delta p_{nm}(t) - \eta_{pk}(t)] \right]. \tag{10}\]

where the symbols \(\alpha_k, \beta_k, \gamma_k, \) and \(\sigma_k\) are given as follows:
\[
\alpha_k = C_{pcnk}N_k \left[ \frac{p_{1k}}{p_{po} + p_{1k}} + \frac{p_{po} + p_{1k}}{N_k} + \frac{\eta_{pk}(t_{i_k})\Delta p(0)}{N_k} \right], \tag{11a}\]
\[
\beta_k = C_{pcnk}N_k \left[ \frac{p_{po} + p_{1k}}{N_k} + \frac{\eta_{pk}(t_{i_k})\Delta n(0)}{N_k} \right], \tag{11b}\]
\[
\gamma_k = C_{ncnk}N_k \left[ \frac{n_{1k}}{n_{po} + n_{1k}} + \frac{n_{po} + n_{1k}}{N_k} + \frac{\eta_{nk}(t_{i_k})\Delta n(0)}{N_k} \right], \tag{11c}\]
\[
\sigma_k = C_{ncnk}N_k \left[ \frac{n_{po} + n_{1k}}{N_k} + \frac{\eta_{nk}(t_{i_k})\Delta p(0)}{N_k} \right]. \tag{11d}\]

Inclusion of the \(\eta_{nk}(t)\) and \(\eta_{pk}(t)\) terms in Eqs. (9) and (10), respectively, is a construction which allows the application of boundary conditions to evaluate the nonlinear terms at \(t=t_0\) in Eqs. (9) and (10). Constants \(\eta_{nk}(t_0)\) and \(\eta_{pk}(t_0)\) are dependent on the injection level. If the resulting Eqs. (9) and (10) are multiplied out, the \(\eta_{nk}(t)\) and \(\eta_{pk}(t)\) terms cancel. Effectively the equations are unchanged by inclusion of these terms.
Two conditions exist for the nonlinear terms to vanish (go to zero) in Eqs. (9) and (10) at a certain time \( t_0 \). For example, in the case of three quantum levels \((m=3)\) the charge neutrality relation Eq. (5) is written as Eq. (12) with two occupancy functions \( \Delta f_1(t_1) \) and \( \Delta f_2(t_2) \) intersecting with \( \Delta n_{nm}(t) \). Note that for a light impulse \( N_a \delta(t) \), carriers are generated with \( \Delta n(0)=\Delta p(0) \) and, \( \Delta n_{nm}(t) \) and \( \Delta P_{nm}(t) \) are the normalized forms of \( \Delta n(t) \) and \( \Delta p(t) \), respectively.

\[
\Delta p_{nm}(t_0) = \Delta n_{nm}(t_0) + \frac{N_1}{\Delta n(0)} \Delta f_1(t_0) + \frac{N_2}{\Delta n(0)} \Delta f_2(t_0) + \frac{N_3}{\Delta n(0)} \Delta f_3(t_0). \tag{12}
\]

At \( t=t_0 \), the following constants are defined:

\[
\eta_{n1}(t_0) = \eta_{n1}(t_1) + \eta_{n2}(t_2), \tag{13a}
\]
\[
\eta_{p1}(t_0) = \eta_{p1}(t_1) + \eta_{p2}(t_2), \tag{13b}
\]

with corresponding terms

\[
\eta_{n1}(t_1) = \phi_1 \eta_{n1}(t_1), \tag{14a}
\]
\[
\eta_{p1}(t_2) = \phi_2 \eta_{n2}(t_2), \tag{14b}
\]

and \( \phi_k \) given by Eq. (15) defined with, for example, \( k=1 \), i.e.,

\[
\phi_1 = \left[ 1 + \frac{N_1}{\Delta n(0)} \right]. \tag{15}
\]

Now from Eqs. (13b) and (14)

\[
\eta_{pk}(t_0) = \phi_1 \eta_{n1}(t_1) + \phi_2 \eta_{n2}(t_2), \tag{16}
\]

so that the charge neutrality relation Eqs. (12) and (16) may be written as Eq. (17), where the constants \( \eta_{pk}(t_0) \) and \( \eta_{nk}(t_0) \) are defined by \( \eta_{pk}(t_0)=\Delta p_{nm}(t_0) \) and \( \eta_{nk}(t_0)=\Delta n_{nm}(t_0) \), respectively.

\[
\Delta p_{nm}(t_0) = \eta_{nk}(t_0) + \frac{N_1}{\Delta n(0)} \Delta f_1(t_2). \tag{17}
\]

Therefore \( \Delta f_1(t_2)=0 \) and this can only occur if the condition \( t_2=0^+ \) for which \( f_1(0^+)=f_1(0^-) \) for \( k=3 \). The other conditions are \( t_1>0^+ \) and \( t_2>0^+ \) for which \( \Delta f_1(t_1)>0 \) and \( \Delta f_2(t_2)>0 \), being the points of intersection with \( \Delta n_{nm}(t) \) at \( t=t_0 \). Hence constants \( \eta_{n1}(t_1)>0 \) and \( \eta_{n2}(t_2)>0 \) and similarly \( \eta_{pk}(t_ik)>0 \) for \( k=1,2 \). The constant \( \eta_{n3}(t_3)=0 \) and similarly \( \eta_{pk}(t_ik)=0 \) for \( k=3 \).

The linear (constant) region of frequency variation with excess carrier concentration corresponds to the case of \( \eta_{nk}(0^)=0 \) and \( \eta_{pk}(0^-)=0 \) for all \( k \). When \( \eta_{pk}(t_ik) > 0^+ \) for some \( k \), this corresponds to the nonlinear region of frequency variation with excess carrier concentration (see Sec. III). Effectively at higher excess carrier concentrations there is an intersection of \( \Delta f_k(t) \) and \( \Delta n_{nm}(t) \), whereas at low injection there is no intersection.

The general equations that allow the determination of the fundamental frequencies \( \omega \) are given by Eqs. (18). From Eqs. (9) and (10) the nonlinear terms vanish at \( t=t_0 \), yielding Eqs. (18a) and (18b). Equation (18c) is derived in Appendix B. See Appendix C for the calculation of \( \eta_{nk}(t_ik) \) and \( \eta_{pk}(t_ik) \) for some \( k \).

\[
- \frac{d\Delta n_{nm}(t)}{dt} \bigg|_{t=t_0} = \sum_{k=1}^{m} \left[ \Delta n_{nm}(t_0) \left( \gamma_k - \sigma_k \right) - \frac{\sigma_k}{\Delta n(0)} N_k \Delta f_k(t_0) \right], \tag{18a}
\]
\[
- \frac{d\Delta P_{nm}(t)}{dt} \bigg|_{t=t_0} = \sum_{k=1}^{m} \left[ \Delta P_{nm}(t_0) \left( \alpha_k - \beta_k \right) + \frac{\beta_k}{\Delta n(0)} N_k \Delta f_k(t_0) \right], \tag{18b}
\]

and for each level \( k \) from Appendix B, Eq. (B18).

\[
\frac{N_k}{\Delta n(0)} \frac{df_k(t)}{dt} \bigg|_{t=t_0} = \left[ \Delta n_{nm}(t_0) \left( \gamma_k - \sigma_k \right) - \frac{\sigma_k}{\Delta n(0)} N_k \Delta f_k(t_0) \right] - \left[ \Delta P_{nm}(t_0) \left( \alpha_k - \beta_k \right) + \frac{\beta_k}{\Delta n(0)} N_k \Delta f_k(t_0) \right] + \sum_{k'=1;k'\neq k}^{m} N_{k'} \frac{N_k}{\Delta n(0)} \left[ C_{nk'k} k_{k'}^n + C_{pk'k} k_{k'}^p \right] \Delta f_k(t_0) + \left[ C_{nk'k} k_{k'}^n + C_{pk'k} k_{k'}^p \right] \Delta f_k(t_0), \tag{18c}
\]

where in Eq. (18c), the symbols \( k_{k'}^{n,p} \), \( k_{k'}^{n,p} \), \( k_{k'}^{n,p} \), and \( k_{k'}^{n,p} \) are defined as Eqs. (B16) in Appendix B.

### III. MULTIPLE DISCRETE DEFECT LEVELS: IDEAL IMPULSE \( \delta(t) \)

Consider the case of three discrete defect energy levels in the band gap where the processes of recombination and emission via each level proceed with the respective band and transitions between levels are not considered. In Refs. 7 and 8 the rate equations are given without formal derivation as a summation of the SRH single-level rate equations. It is shown in Appendix A that this formulation is effectively correct although one cannot, strictly speaking, define individual level carrier recombination rates. There is only one recombination rate for each carrier type. With reference to Fig. 1 for three defect levels \((n=3)\) the following rate equations (19) apply for recombination via and emission from the \( k \)th defect level, where \( k=1,2,3 \). A set of linear equations (19) is formed from Eqs. (5) and (18) with the differential operator \( D=d/dt \), where Eq. (18b) is dependent and not included. The terms \( N_k/\Delta n(0) \) in the respective columns cancel.
and there intersection between \( D = 0 \):

\[
D + \sum_{k=1}^{m=3} (\gamma - \sigma_k)_{k} 0 \quad -\sigma_1 \frac{N_1}{\Delta n(0)} -\sigma_2 \frac{N_2}{\Delta n(0)} -\sigma_3 \frac{N_3}{\Delta n(0)} \\
- (\gamma - \sigma)_1 (\alpha - \beta)_1 [D + (\sigma + \beta)_1] 0 0 \\
\times \frac{N_1}{\Delta n(0)} \\
- (\gamma - \sigma)_2 (\alpha - \beta)_2 0 [D + (\sigma + \beta)_2] 0 \\
\times \frac{N_2}{\Delta n(0)} \\
- (\gamma - \sigma)_3 (\alpha - \beta)_3 0 0 [D + (\sigma + \beta)_3] \\
\times \frac{N_3}{\Delta n(0)} \\
1 -1 \quad \frac{N_1}{\Delta n(0)} \quad \frac{N_2}{\Delta n(0)} \quad \frac{N_3}{\Delta n(0)}
\]

where in order to simplify notation

\[
(\gamma - \sigma_k) = (\gamma - \sigma)_k, \quad (\alpha - \beta_k) = (\alpha - \beta)_k, \quad (\sigma + \beta_k) = (\sigma + \beta)_k. 
\]

The characteristic equation is formed from the determinant of the matrix Eq. (19). From the characteristic equation a polynomial in \( \omega \) is evaluated for the \( m + 1 \) eigenvalues or fundamental frequencies. The linear (constant) region of frequency variation with excess carrier concentration is determined by differentiating frequency \( \omega_{1+k} \) with respect to excess carrier concentration and equating the result to zero.

The method using the differential operator \( D \) of Refs. 11 and 12 whereby the determinant of the matrix of Eqs. (19) equals zero, allows the formation of the characteristic equation:

\[
(D^4 + bD^3 + cD^2 + dD + e)\exp(-\omega t) = 0, 
\]

where

\[
\omega^4 - b\omega^3 + c\omega^2 - d\omega + e = 0. 
\]

The fundamental frequencies \( \omega_{1,2,3,4} \) may be evaluated by solving for the four roots of Eq. (22) with the frequencies given by \( \omega_1 = 1/\tau_1, \omega_2 = 1/\tau_2, \omega_3 = 1/\tau_3, \) and \( \omega_4 = 1/\tau_4. \)

The linear region (constant region) of variation of fundamental frequencies \( \omega_{1,2,3,4} \) with injection level or excess carrier density is defined by Eq. (23). This condition is satisfied upon expansion of Eq. (23) for \( t_2 = 0^+ \) with \( \eta_{jk}(t_2) = 0 \) and \( \eta_{jk}(t_2) = 0. \) This corresponds to the situation where there is no intersection between \( \Delta f_j(t_0) \) and \( \Delta n_{nm}(t_0) \) at one or more \( t = t_{ik} \) for \( k = 1,2, \ldots, m: \)

\[
\left| \frac{d\omega_{1,2,3,4}}{d\Delta n(0)} \right| > 0. 
\]

Having evaluated the fundamental frequencies for the multiple discrete state system, an analytic solution of the differential rate equations for \( \Delta n(t) \) and \( \Delta p(t) \) may be written as Eqs. (25) and (26), respectively, i.e.,

\[
\Delta n(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{u=0}^{\infty} N_{i,j,u,v} \times \exp(-i\omega_{1} t - j\omega_{2} t - u\omega_{3} t - v\omega_{4} t), \quad (25)
\]

\[
\Delta p(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{u=0}^{\infty} P_{i,j,u,v} \times \exp(-i\omega_{1} t - j\omega_{2} t - u\omega_{3} t - v\omega_{4} t). \quad (26)
\]

IV. MULTIPLE EXCITED STATES: IDEAL IMPULSE 

For the excited states system with \( m = 2 \) localized states (see Figs. 1 and 2), the fundamental frequencies or eigenvalues are determined from the linear equations formed from...
Eqs. (5) and (18). From these equations the determinant of the set of linear equations (at \( t=t_0 \)) is expressed as a matrix given by Eq. (27) using the notation of Eq. (20) with the differential operator \( D=d/dt \). Again the terms \( N_1/\Delta n(0) \) (in the third column) and \( N_2/\Delta n(0) \) (in the fourth column), not included in the columns of the determinant, cancel. The equation for \( d\Delta p_{nm}(t)/dt \) for the respective \( k \) is not included in the determinant as it is dependent:

\[
\begin{vmatrix}
  m=2 & D + \sum_{k=1}^{m} (\gamma - \alpha) \frac{D}{D} & 0 & -\alpha_1 & -\alpha_2 \\
 1 & -(\gamma - \alpha)_1 & (\alpha - \beta)_1 & [D + (\sigma + \beta)_1] & -N_1 [C_f o_1 + C_{n_1} (1 - f o_1)] \\
 1 & -(\gamma - \alpha)_2 & (\alpha - \beta)_2 & -N_2 [C_p o_2 + C_{p_2} (1 - f o_2)] & N_2 [C_n o_2 + C_{n_2} (1 - f o_2)] \\
 1 & 1 & -1 & 1 & [D + (\alpha + \beta)_3] \\
 1 & 1 & 1 & 1 & +N_1 [C_n o_2 + C_{n_2} (1 - f o_2)] \\
\end{vmatrix} = 0.
\]

The method using the differential operator \( D \) of Refs. 11 and 12 whereby the determinant Eq. (27) equals zero, allows the formation of the characteristic equation:

\[
(D^3 + bD^2 + cD + d)\exp(-\omega t) = 0,
\]

where

\[-\omega^3 + b\omega^2 + c\omega + d = 0.
\]

The fundamental frequencies \( \omega_{1,2,3} \) may be solved for by solving for the three roots of Eq. (29) with the fundamental frequencies given by \( \omega_1=1/\tau_1, \omega_2=1/\tau_2, \) and \( \omega_3=1/\tau_3. \)

The linear region (constant region) of variation of fundamental frequencies \( \omega_{1,2,3} \) with injection level or excess carrier density is defined by Eq. (30). This condition is satisfied upon expansion of Eq. (30) for \( t_0=0^+ \) with \( \eta_{nk}(t_0)=0 \) and \( \eta_{nk}(t_0)=0. \) This corresponds to the situation where there is no intersection between the \( \Delta f_k(t_0) \) and \( \Delta n_{nm}(t_0). \) An upper limit for the linear region of excess carrier density exists, referred to as \( \Delta n(0)_{\text{critical}} \), and is evaluated in Appendix C. Hence, Eqs. (6) and (7) are linear for all injection levels less than \( \Delta n(0)_{\text{critical}} \) for the given conditions, \( \eta_{nk}(t_0)=0 \) and \( \eta_{nk}(t_0)=0. \)

\[
\frac{d\omega_{1,2,3}}{d\Delta n(0)} > 0.
\]

The nonlinear region of variation of fundamental frequencies \( \omega_{1,2,3} \) with injection level or excess carrier density is defined by Eq. (31). This corresponds to the range of excess carrier density whereby the change in fundamental frequency with injection level is nonzero as expressed by Eq. (31). This condition is satisfied for \( \eta_{nk}(t_0)>0 \) and \( \eta_{nk}(t_0)>0. \) For some value of \( t_0, \) and corresponds to an intersection of one or more of the \( \Delta f_k(t) \) with \( \Delta n_{nm}(t) \) and each intersection at some \( t=t_0: \)

\[
\frac{d\omega_{1,2,3}}{d\Delta n(0)} > 0.
\]

Having evaluated the fundamental frequencies for the excited state system, an analytic solution of the differential rate equations for \( \Delta n(t) \) and \( \Delta p(t) \) may be written as Eqs. (32) and (33), respectively,

\[
\Delta n(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} N_{i,j,k} \exp(-i\omega_i t - j\omega_j t - u\omega_k t),
\]

\[
\Delta p(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} P_{i,j,k} \exp(-i\omega_i t - j\omega_j t - u\omega_k t).
\]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Level one</th>
<th>Level two</th>
<th>Level three</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_i - E_c ) energy depth from valence band [eV]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( N_i ) ((k=1,2,3)) discrete level concentration ([\text{cm}^{-3}])</td>
<td>(5 \times 10^{12})</td>
<td>(5 \times 10^{11})</td>
<td>(5 \times 10^{10})</td>
</tr>
<tr>
<td>( \sigma_{pk} ) ((k=1,2,3)) cross section for holes ([\text{cm}^2])</td>
<td>(8.5 \times 10^{-14})</td>
<td>(8.5 \times 10^{-17})</td>
<td>(8.5 \times 10^{-17})</td>
</tr>
<tr>
<td>( \sigma_{ek} ) ((k=1,2,3)) cross section for electrons ([\text{cm}^2])</td>
<td>(9.0 \times 10^{-17})</td>
<td>(9.0 \times 10^{-15})</td>
<td>(9.0 \times 10^{-14})</td>
</tr>
</tbody>
</table>
\[ \Delta p(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{u=0}^{\infty} P_{i,j,u} \exp(-i\omega_1 t - j\omega_2 t - u\omega_3 t). \] (33)

V. NUMERICAL SOLUTION FOR MULTIPLE LOCALIZED STATES: IDEAL IMPULSE \( \delta(t) \)

A fifth order Runge-Kutta scheme, as in Refs. 13 and 14, is chosen to represent discrete values of \( \Delta n(t+\Delta t), \Delta p(t+\Delta t), \) and \( \Delta f_k(t+\Delta t) \) to ensure convergence to the equilibrium concentration for the minority carrier as \( t \to \infty \). The normalized excess carrier concentrations at \( t=0^+ \) are given by \( \Delta n_{nm}(0^+)=1.0 \) and \( \Delta p_{nm}(0^+)=1.0 \). The defect level initial occupation is assumed not to change from \( t=0^- \) to \( t=0^+ \). Tables I and II indicate typical data used in the numerical solution for the discrete states system and the excited states system, respectively.

Multitransient analysis of the numerical solution of the given equations results in the determination of frequencies which are compared with the derived theoretical expressions represented by Eq. (22). Multitransient analysis refers to the extraction of component exponential terms from a sum of exponentials comprising the signal. The methods discussed in Refs. 15–17 present a means of analyzing signals with additive white Gaussian noise. In addition where the data matrix and the observation matrix may be subject to noise fluctuations, the total least squares method (TLS) of Ref. 18 is employed. This method models the multieponential signal as an autoregressive process. For the present analysis the TLS multitransient analysis method of Ref. 17 using singular value decomposition\(^{19}\) is applied to the analysis of the multicomponent exponential decay.

The magnitude, of the coefficients \( N_{i,j,u,v} \) and \( P_{i,j,u,v} \) of the exponential terms, falls away rapidly to a very small value so that less than eight terms are significant in the series. The first two coefficients are the most significant, \( (i,j,u,v)=(1,0,0,0) \) and \( (0,1,0,0) \). Expressions for the minority carrier time constant \( \tau_i \) and majority time constants \( \tau_{1+k} \) are derived from Eq. (22) and (29) for the discrete and coupled states, respectively. The dominant time constant is \( \tau_1 \) and its behavior with excess carrier concentration indicates the onset of trapping. A critical point, representing the transition between the linear (constant) and nonlinear variation of fundamental frequency with excess carrier density, is identified.

A. Multiple discrete states

Considering the multiple discrete state system the differential equations are of the form (34)–(36) taken from Eqs. (1)–(3). There is no interaction between levels so that the terms \( u_{k,k'}^{np} \) and \( u_{k,k'}^{pp} \) go to zero in Eq. (3) (see Appendix B):

\[ \frac{dn_k(t)}{dt} = \sum_{k=1}^{3} \left\{ e_{nke} N_{f_k}(t) - C_{nck} N_k n(t)[1 - f_k(t)] \right\}, \] (34)

\[ \frac{dp_k(t)}{dt} = \sum_{k=1}^{3} \left\{ e_{pke} N_{f_k}(t) - C_{pck} N_k p(t)f_k(t) \right\}, \] (35)

and

\[ N_k \frac{df_k(t)}{dt} = - \left\{ e_{nke} N_{f_k}(t) - C_{nck} N_k n(t)[1 - f_k(t)] \right\} \]

\[ + \left\{ e_{pke} N_{f_k}(t) - C_{pck} N_k p(t)f_k(t) \right\}. \] (36)

Equations (34)–(36) form a set of coupled differential equations which are nonlinear and solved numerically. A correlation between the numerical solution and the predictions of the analytical solution is made and found to indicate a close agreement.

TLS multitransient analysis of the numerical solution of Eqs. (34)–(36) with \( m=3 \) results in the determination of frequencies which are compared with those derived from the theoretical expression Eq. (22). Figure 3 shows the comparison between the decay time constants \( \tau_{1+k} = 1/\omega_{1+k} \) for \( m=3 \) and \( k=1,2,3 \), as calculated from Eq. (22) and as evaluated by TLS analysis of numerical data \( \Delta n(t+\Delta t) \) for \( T = 300 \) K. The figure indicates the variation of fundamental

![FIG. 3. A graph of \( \tau_1 = \omega_1, \tau_2 = \omega_2, \tau_3 = \omega_3 \) and \( \tau_4 = \omega_4 \) vs excess electron concentration \( \Delta n(0) \) for the multiple discrete level system. The doping concentration \( N_d \) of the sample is \( 5 \times 10^{14} \) cm\(^{-3} \). See Table I for defect level parameters. Continuous line: \( \tau_1,\tau_2,\tau_3,\tau_4 \) predicted by Eq. (22). Circles: \( \tau_1,\tau_2,\tau_3,\tau_4 \) derived from multitransient analysis of the numerical solution.](Image)
frequencies with excess carrier concentration. The agreement is excellent. The onset of the nonlinear variation of frequencies \( \omega_{1+k} \) with \( \Delta n(0) \) for \( \Delta n(0) = \Delta n(0)_{\text{critical}} \) is clearly seen in the figure, where the linear portion extends to \( 5 \times 10^{11} \ \text{cm}^{-3} \) and begins to diverge for \( \Delta n(0) \) greater than \( 5 \times 10^{11} \ \text{cm}^{-3} \). This indicates that the linear region is in close agreement with the definition given by Eq. (23). See Appendix C for the evaluation of \( \Delta n(0)_{\text{critical}} \). For parameters listed in Table I, \( \Delta n(0)_{\text{critical}} = 5.44 \times 10^{13} \ \text{cm}^{-3} \). Also note that TLS multitransient analysis of the numerical solution results in \( \tau_{1+k} \), in close agreement with that predicted by the solution for \( \tau_{1+k} \) of Eq. (22) for high injection. Some values (circles) could not be resolved due to the very small magnitude of the coefficient of the exponential term.

Figure 4 shows the comparison between the decay time constants \( \tau_{1,2,3,4} \), as calculated from Eq. (22) for \( T = 300 \ \text{K} \), versus the Fermi level \( E_F \). The doping concentration for the p-type and n-type samples varies from \( 5 \times 10^{10} \ \text{cm}^{-3} \) to \( 1 \times 10^{16} \ \text{cm}^{-3} \). Agreement is very good between the analysis of the numerical solution and the calculated \( \tau_{1+k} \).

Figure 3 indicates that for \( \Delta n(0) > \Delta n(0)_{\text{critical}} \), \( \tau_{1} \) increases realizing a situation where the minority carrier (p type) time constant represented by \( \tau_{1} \) is becoming longer. The minority carrier electrons are remaining in the conduction band for a longer average time indicating that there are fewer holes at the defect levels to recombine with. As such the defect level electron population is increasing and electrons are remaining at the defect levels for a longer average time. This is referred to as minority carrier trapping. The time constant \( \tau_{1} \) remains constant in the linear region of variation of \( \tau_{1+k} \) with \( \Delta n(0) \). This region corresponds to recombination only whereas the nonlinear region undergoes recombination and trapping. Eventually the trapping saturates as \( \Delta n(0) \) is increased to high injection and the time constant \( \tau_{1} \) reaches a maximum becoming almost constant. From this figure \( \tau_{1} \) describes the minority carrier decay constant and \( \tau_{1+k} \) with \( k \gg 1 \) describes the major carrier decay constants associated with each level. Hence, they represent fundamental decay time constants.

**TABLE III.** TLS analysis of numerical data \( \Delta n(t + \Delta t) \) for the case of level one, level two and level three included in the model. Evaluation of Eqs. (22) and (25) for \( N_d = 5 \times 10^{13} \ \text{cm}^{-3} \) with \( \Delta n(0) = 5 \times 10^{10} \ \text{cm}^{-3} \) (low injection).

<table>
<thead>
<tr>
<th>Fundamental</th>
<th>Equation (22)</th>
<th>TLS analysis</th>
<th>TLS coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>frequency ( \omega )</td>
<td>( \omega ) (s(^{-1}))</td>
<td>( \omega ) (s(^{-1}))</td>
<td>numerical</td>
</tr>
<tr>
<td>( \omega_1 )</td>
<td>4.524 ( 10^{4} )</td>
<td>4.523 ( 10^{4} )</td>
<td>( N_{1000} = 6.277 \times 10^{-1} )</td>
</tr>
<tr>
<td>( 2 \omega_1 )</td>
<td>8.785 ( 10^{4} )</td>
<td>6.722 ( 10^{4} )</td>
<td>( N_{1000} = 6.277 \times 10^{-4} )</td>
</tr>
<tr>
<td>( \omega_2 )</td>
<td>19.418 ( 10^{4} )</td>
<td>19.406 ( 10^{4} )</td>
<td>( N_{1000} = 6.277 \times 10^{-4} )</td>
</tr>
<tr>
<td>( \omega_3 )</td>
<td>22.470 ( 10^{4} )</td>
<td>22.470 ( 10^{4} )</td>
<td>( N_{1000} = 6.277 \times 10^{-4} )</td>
</tr>
<tr>
<td>( \omega_4 )</td>
<td>38.376 ( 10^{4} )</td>
<td>38.376 ( 10^{4} )</td>
<td>( N_{1000} = 6.277 \times 10^{-4} )</td>
</tr>
<tr>
<td>( \omega_5 )</td>
<td>50.622 ( 10^{4} )</td>
<td>50.622 ( 10^{4} )</td>
<td>( N_{1000} = 6.277 \times 10^{-4} )</td>
</tr>
<tr>
<td>( \omega_6 )</td>
<td>677.381 ( 10^{4} )</td>
<td>677.381 ( 10^{4} )</td>
<td>( N_{1000} = 6.277 \times 10^{-4} )</td>
</tr>
<tr>
<td>( \omega_7 )</td>
<td>1989.850 ( 10^{4} )</td>
<td>1989.850 ( 10^{4} )</td>
<td>( N_{1000} = 6.277 \times 10^{-4} )</td>
</tr>
</tbody>
</table>

Further by recalling the expression Eq. (25) for the decay \( \Delta n_{n+1}(t) \) the infinity of time constants is found to be linear combinations of \( m+1 \) fundamental frequencies \( \omega_{1+k} \). Table III indicates the correlation with the prediction of Eqs. (22) and (25) and TLS analysis of the decay \( \Delta n(t + \Delta t) \) within the limitations of the TLS method. No \textit{a priori} knowledge is available for the time constants associated with the numerical solution, yet the agreement of the analytical and TLS methods with respect to the linear combination of \( \omega_{1+k} \) is excellent. Note that the magnitudes of the coefficients decrease rapidly. In addition for the transient case in low injection, the magnitude of the coefficient for \( \tau_2 \) is significant.

**B. Multiple excited states**

The equations describing the time evolution of the decay for the ground excited states system \( m=2 \), are given by Eqs. (34), (35), and (37) [see Appendix B for Eq. (37)].

Further by recalling the expression Eq. (25) for the decay \( \Delta n_{n+1}(t) \) the infinity of time constants is found to be linear combinations of \( m+1 \) fundamental frequencies \( \omega_{1+k} \). Table III indicates the correlation with the prediction of Eqs. (22) and (25) and TLS analysis of the decay \( \Delta n(t + \Delta t) \) within the limitations of the TLS method. No \textit{a priori} knowledge is available for the time constants associated with the numerical solution, yet the agreement of the analytical and TLS methods with respect to the linear combination of \( \omega_{1+k} \) is excellent. Note that the magnitudes of the coefficients decrease rapidly. In addition for the transient case in low injection, the magnitude of the coefficient for \( \tau_2 \) is significant.
Figure 5. A graph of $\tau_1=1/\omega_1$, $\tau_2=1/\omega_2$, and $\tau_3=1/\omega_3$, for the excited state, vs excess electron concentration $\Delta n(0)$. The doping concentration $N_D$ of the sample is $5 \times 10^{13}$ cm$^{-3}$. See Table II for defect level parameters. Continuous line: $\tau_1, \tau_2, \tau_3$ predicted by Eq. (29). Circles: $\tau_1, \tau_2, \tau_3$ derived from multitransient analysis of the numerical solution.

Figure 6 shows the comparison between the decay time constants $\tau_1$, $\tau_2$, and $\tau_3$, as calculated from Eq. (29) for $T=300$ K, versus the Fermi level $E_F$. The doping concentration for the $p$-type and $n$-type samples varies from $5 \times 10^{10}$ cm$^{-3}$ to $1 \times 10^{16}$ cm$^{-3}$. Agreement is very good for the numerical solution and the calculated $\tau_{1,2,3}$. From Fig. 5 and 6 $\tau_1$ describes the minority carrier decay constant and $\tau_{2,3}$ describes the majority carrier decay constants associated with the two levels.

VI. EXPERIMENTAL CONSTRAINTS

A $P+ + N$-Czochralski-grown silicon pn junction sample in conjunction with the deep level transient spectroscopy (DLT) measurement apparatus serves the purpose of demonstrating the theory presented in this work. In the temperature range 100 K–310 K capacitance versus time records are taken at approximately single degree temperature increments. Each capacitance versus time record, sampled at $2.5 \mu$s with 4096 samples per record, is analyzed using a multitransient technique for component time constants. Averaging of the sampled records reduces the effect of the noise. This improvement in the signal-to-noise ratio may be evaluated by determining the standard deviation of the noise as in Refs. 20 and 21. On the low doped $N$-side of the junction the electric field is assumed to be below the critical value where the Poole-Frenkel effect may enhance the emission of carriers.

In order to effect measurements in the linear region (constant region) of $\tau$ versus excess carrier concentration, a low excess carrier concentration is required and a degree of sensitivity in detecting the change in the multiple-level occupancy. However, from the theory presented, the excited states system even for low concentrations of defects yields a long fundamental time constant $\tau_1$ (see also Ref. 22) for an excess carrier concentration above the critical value which is relatively low (see Fig. 5). In the nonlinear region the fundamental time constants vary with excess carrier concentration. Effectively the DLTS pulse may be injecting carriers in the nonlinear region of $\tau$ versus excess carrier concentration for the excited state system. Also, the capture cross section may vary with temperature whereas for simplicity in the theory presented the cross sections are constant with temperature.

Metalization is applied to the front and back surfaces for the contacts without a guard ring. It is assumed that the cleaved surfaces of the die provide a high interface or surface state density so that the Fermi level is effectively pinned at the unmetalized surfaces (cleaved surfaces) at the sides of the die. It is assumed that there is no carrier injection from these interface states into the bulk.

VII. EXPERIMENTAL CONDITIONS

A $Pn$ junction is formed in the Czochralski-grown silicon sample by a boron diffused $P++$ layer, with a concentration of $1 \times 10^{20}$ cm$^{-3}$, on a $N$-substrate with a concentration of $1.82 \times 10^{14}$ cm$^{-3}$ measured by the CV technique. The width of the $P++$ layer is typically $10–20$ $\mu$m and the $N$-substrate 400 $\mu$m. The defects were intentionally introduced by neutron irradiation with a fluence of $\approx 2 \times 10^{15}$ cm$^{-2}$. The die size of the sample is $2 \times 2$ mm$^2$ (0.04 cm$^2$) with sintered metal contacts over the front and back surfaces.

The voltage pulse applied to the $N$-side of the $pn$ junction allowed for 0.0 V forward and −0.5 V reverse with a pulse width of $100 \mu$s and a period width of 10.24 ms. Sampled records at each temperature were averaged over 1 s to reduce the noise. Effectively the records were averaged $\approx 100$ times. This reduced the standard deviation of the noise to $\approx 10^{-11}$, which is consistent with that of the numerical analysis ($10^{-18}$). At least six significant digits are required in the mantissa of the data sets. Furthermore, a digital filter after the capacittance stage is bypassed. This filter may produce some smoothing of the data and introduce a correlation between the samples. This smoothing is undesirable in terms of the multitransient analysis for the faster time constants.

The averaged records of the capacitance transients are recorded and analyzed offline. Figure 7 shows the results of the multitransient analysis for the component time constants.
versus $1/kT$ for the range of temperatures 100 K–300 K. Table IV indicates typical fundamental frequencies for the given temperatures. A limitation on resolving the faster time constants is the response time of 2 μs of the capacitance bridge.

VIII. DISCUSSION

From the theory presented it is evident that $\tau_1$ represents an important parameter in terms of the minority carrier trapping behavior. This extends to both the multiple discrete defect level system and the ground excited state defect level system. These two systems are considered to be the most significant sources of defect level effects on the minority carrier. In particular the linear region which represents recombination only allows comparison of semiconductor time constants ($\tau_1$). Comparison of time constants in the nonlinear region is at best unreliable as the degree of trapping is unknown and the range of variation of $\tau_1$ with excess carrier concentration is significant.

It is also evident that even a very low concentration of ground excited state defect levels (see Table II) has a dramatic effect on the time constant $\tau_1$. This in itself indicates the sensitivity of the fundamental time constants to a range of defect level concentrations from the coupled-level system to the discrete-level system. Also, for such a system the effect is significantly different for p type as opposed to n type (see Fig. 6). Again a measurement of $\tau_1$ from the linear to the nonlinear regions indicates a recombination-only region and a trapping region (see Fig. 5).

The fundamental frequencies determined from experiment, and plotted in Fig. 7, indicate consistent support for the theory presented in this work. Typical values are given in Table IV and from the number of fundamental frequencies this would indicate three defect levels.

The proposition that the multiple levels are independent is supported by the theory that the resultant inverse decay time constant is equal to the sum or linear combination of the individual inverse decay constants or level emission rates. For multiple discrete levels the fundamental frequencies are nonlinear functions of the capture and emission rates as shown by Eq. (22) where the roots of this polynomial equation represent the fundamental frequencies. The coefficients of the polynomial are formed from the matrix given by Eq. (19) and immediately it is evident that the relationship between the fundamental frequencies is a complex one (see also Eq. (32) of Ref. 4 for the single level). Also, this further demonstrates that discrete levels and coupled levels are not independent as the resultant inverse decay constant cannot be represented by the sum of the individual inverse decay constants or in simple terms the sum of the emission rates.

IX. CONCLUSIONS

Using the method of solution developed in Ref. 4 it is shown that the fundamental frequencies determined from the derived differential rate equations for the two defect level systems are consistent with the numerical solution of the rate equations. The linear form of the rate equations predicts fundamental time constants which agree with those determined from the analysis of the numerical solution. The analytic solution to the differential rate equations for the multiple
defect state systems indicates an infinity of exponential terms, the frequencies of which are a linear combination of fundamental frequencies. The fundamental frequencies are derived from the linear form of the rate equations without an approximation at a given temperature for arbitrary, excess carrier concentration, doping concentration $N_{A,D}$, defect level concentration $N_k$, cross section $\sigma_{n,p}$ and energy level $E_k$. The transition point from the linear (constant) to nonlinear variation of fundamental frequency with excess carrier concentration is also given. Trapping behavior of the minority carrier transition point from the linear form of the rate equations without an approximation at a given temperature for arbitrary, excess carrier concentration, doping concentration $N_{A,D}$, defect level concentration $N_k$, cross section $\sigma_{n,p}$ and energy level $E_k$. The transition point from the linear (constant) to nonlinear variation of fundamental frequency with excess carrier concentration is also given. Trapping behavior of the minority carrier decay time constants are given. There is no a priori knowledge of the number and value of time constants in the numerical solution. This independent verification of the theory, determined from the first principles derivation of the rate equations, confirms the properties of the above solution and the method of multitransient analysis.

ACKNOWLEDGMENTS

The Special Research Center of Excellence for Advanced Silicon Photovoltaics and Photonics is supported by the Australian Research Council’s Special Research Centers scheme. The author extends his appreciation to Dr. G. Webster of Accent Optical Technologies Inc. for DLTS measurements and data on the given sample.

APPENDIX A: GENERAL RATE EQUATIONS FOR RECOMBINATION BETWEEN THE CONDUCTION BAND AND LOCALIZED STATES

A first principles approach is adopted as in Ref. 1 for deriving the differential rate equations for carrier transitions via multiple discrete defect levels in the band gap. The recombination rate for electrons at a discrete defect energy level is the difference in the rate of electron capture and the rate of electron emission. Consider $m$ discrete defect levels where there is no carrier exchange between levels. Furthermore, a carrier at the $k'$ level in the range of energy $dE_{k'}$, in the conduction band, makes a transition to the $k$th discrete defect level as shown in Fig. 8. The rate of electron capture from the conduction band to multiple levels may be expressed as being proportional to the number of electrons in the conduction band $f(E_k)N(E_k)dE_k$ [where $f(E_k)$ is the electron occupation probability function] and the number of holes $N_k[1-f_s(E_k)]$ at the $k$th defect level, where $f_s(E_k)$ is the electron defect level occupation function for a given $k$. The constant of proportionality is $c_p(E_k',E_k)$. Similarly the rate of electron emission from the defect levels to the conduction band is proportional to the number of available empty states $f_s(E_k')N(E_k')dE_k'$, so that $f_s(E_k')$ is the electron defect level occupation function for a given $k$. The constant of proportionality is $c_p(E_k',E_k)$. An average capture function $C_{n,k}^p$ of the constant of proportionality $c_p(E_k',E_k)$, is defined over all $k'$ levels in the conduction band for a given localized state $k$. An incremental recombination term $dU_n(E_k',E_k)$ describes the carrier transitions for a $(k',k)$ pair. The total recombination rate $U_n$ given by Eq. (A1) is formed by summing the incremental term over the $k'$ levels and over the $k$ discrete levels. Later, the summation over the continuum of $k'$ states, in the conduction band, is replaced by an integration over these states:

\[ U_n = \sum_{k'=1}^{m} \sum_{E_k'} dU_n(E_k',E_k) \quad (A1) \]

with

\[ dU_n(E_k',E_k) = c_p(E_k',E_k)N_k(1-f_s(E_k))[f(E_k)N(E_k')dE_k' - e_p(E_k,E_k')N_k f_s(E_k)f_p(E_k')N(E_k')dE_k'] \]

where

\[ f_s(E_k') = \frac{1}{1 + \exp\left(\frac{E_k' - E_{Fk}(t)}{kT}\right)} \quad (A3a) \]

\[ f(E_k') = \frac{1}{1 + \exp\left(\frac{E_k' - E_{Fk}(t)}{kT}\right)} \quad (A3b) \]

At equilibrium the recombination rate $dU_n$ is zero and $E_{Fk}(0^+) = E_F$, where $E_{Fk}$ is the $k$th defect level occupancy Fermi level and $E_F$ is the equilibrium Fermi level. The proportionality constant $c_p(E_k',E_k)$ is also nonzero for the same $(k',k)$ pair since the occupancy functions are nonzero at equilibrium. Hence carrier transitions occur in unique pairs $(k',k)$, so that Eq. (A2) is only valid for a corresponding pair to the exclusion of all other pairs. The mathematical equivalent for the discrete case where there is no interaction between levels is that the quantities $f_s(E_k)$ (and hence $[1-f_s(E_k)]$) and $f(E_k')$ (and hence $[1-f(E_k')]$) are orthogonal functions (see Ref. 23) with respect to functions of $E_k$ and $E_k'$ as follows:

\[ dU_n = 0 \quad \text{for} \quad k' \neq k, \quad (A4a) \]

![FIG. 8. Carrier transitions between the $k'$ level in the conduction band and the $k$ level defect state or localized state in the band gap.](image-url)
\[ |dU_n| > 0 \quad \text{for} \quad k' = k. \quad (A4b) \]

From Eq. (A2) the following may be established from the given equilibrium condition and the orthogonal relations Eqs. (A4) for all \((k', k)\) pairs:

\[ e_n(E_{k'}, E_k) = c_n(E_k', E_k) \exp \left( \frac{E_k - E_k'}{kT} \right). \quad (A5) \]

Substitution of Eq. (A5) into Eq. (A2) results in the following:

\[ U_n = \sum_{k=1}^{m} \sum_{E_k' + \Delta E_k'} \left[ 1 - \exp \left( \frac{E_{k'}(t) - E_{k'}(t)}{kT} \right) \right] N_k[1 - f_k(E_k)] \times c_n(E_k', E_k)f(E_k')N(E_{k'})dE_{k'}. \quad (A6) \]

Letting the summation in the conduction band to be an integral over the continuum of \(k'\) levels yields an expression for the total recombination \(U_n\):

\[ U_n = \sum_{k=1}^{m} \left[ 1 - \exp \left( \frac{E_{k'}(t) - E_{k'}(t)}{kT} \right) \right] N_k[1 - f_k(E_k)] \times \int_{E_{k'}}^{E_c} c_n(E_k', E_k)f(E_k')N(E_{k'})dE_{k'}. \quad (A7) \]

The carrier concentration in the conduction band is given by Eq. (A8), using the notation above, and by Eq. (A9) from Fermi-Dirac statistics:

\[ n(t) = \int_{E_{k'}}^{E_c} N(E_{k'})f(E_{k'})dE_{k'}. \quad (A8) \]

\[ n(t) = N_c \exp \left( \frac{E_{k}(t) - E_{k}}{kT} \right). \quad (A9) \]

From the above equations an expression for \(N_c\) is derived given by Eq. (A10) as in Ref. 1, where \(f(E_{k'})\) is written in terms of \(f_p(E_k)\) [See Eq. (A11)],

\[ N_c = \int_{E_{k'}}^{E_c} \left[ \exp \left( \frac{E_k - E_{k'}}{kT} \right) \right] N(E_{k'})dE_{k'}. \quad (A10) \]

where

\[ f(E_{k'}) = \exp \left( \frac{E_k(t) - E_{k'}(t)}{kT} \right) f_p(E_k'). \quad (A11) \]

Note that in the conduction band most of the carriers are near \(E_c\) and so \( f_p(E_k') \) is nearly unity. An average value \(C_{nck}\) for \(c_n(E_k', E_k)\), over all levels \(k'\) for a given \(k\), is defined by Eq. (A12). It is the average value that is measured experimentally:

\[ C_{nck} = \langle c_n(E_k', E_k) \rangle = \int_{E_{k'}}^{E_c} \left[ \exp \left( \frac{E_k - E_{k'}}{kT} \right) \right] c_n(E_k', E_k)N(E_{k'})dE_{k'}. \quad (A12) \]

Hence, the following equation is an equivalent expression for the integrand of Eq. (A7) using the definition \(C_{nck}\) for the average capture coefficient:

\[ \int_{E_{k'}}^{E_c} c_n(E_k', E_k)f(E_k')N(E_{k'})dE_{k'} = \int_{E_{k'}}^{E_c} c_n(E_k', E_k)f(E_k')N(E_{k'})dE_{k'}. \quad (A13) \]

The resulting equation for the recombination rate \(U_n\) is as follows:

\[ U_n = \sum_{k=1}^{m} C_{nck} \left[ 1 - \exp \left( \frac{E_{k'}(t) - E_{k'}(t)}{kT} \right) \right] \times N_k[1 - f_k(E_k)]n(t). \quad (A14) \]

This may be further refined with \(n_{1k}\) given by Eq. (4a):

\[ U_n = \sum_{k=1}^{m} [C_{nck}N_k[1 - f_k(E_k)]n(t) - C_{nck}N_kf_k(E_k)n_{1k}]. \quad (A15) \]

Similarly an expression for \(U_p\) may be derived with \(p_{1k}\) given by Eq. (4b):

\[ U_p = \sum_{k=1}^{m} [C_{pck}N_kf_k(E_k)p(t) - C_{pck}N_k[1 - f_k(E_k)]p_{1k}]. \quad (A16) \]

It may be shown that the Eqs. (9) and (10) may be directly derived from Eqs. (A15) and (A16), respectively. Note that there is only one recombination rate. The summation of \(N_kf_k(E_k)\) over all discrete defect levels is equal to \(\Delta p(t) - \Delta n(t)\) from the charge neutrality relation. From the charge neutrality relation one may write

\[ \left[ \sum_{k=1}^{m} N_k \frac{df_k(E_k)}{dt} \right] = U_n - U_p, \quad (A17) \]

and from the orthogonality condition Eqs. (A4) for each \(k\) the following is established:

\[ \left[ \sum_{k=1}^{m} \frac{df_k(E_k)}{dt} \right] = U_{nk} - U_{pk}, \quad (A18) \]

where the following are defined to simplify the notation and do not indicate a level recombination rate:

\[ U_{nk} = C_{nck}N_k[1 - f_k(E_k)]n(t) - C_{nck}N_kf_k(E_k)n_{1k} \quad (A19) \]

and

\[ U_{pk} = C_{pck}N_kf_k(E_k)p(t) - C_{pck}N_k[1 - f_k(E_k)]p_{1k}. \quad (A20) \]

**APPENDIX B: RATE EQUATIONS FOR LOCALIZED STATE TO LOCALIZED STATE TRANSITIONS**

With reference to Fig. 2 the carrier transitions at the \(k\)th localized state are described by a level occupancy rate as follows:
where Fig. 2 describes electron and hole transitions from the \( k' \) state to the \( k \) state as Eq. (B2a) and the \( k \) state to the \( k' \) state as Eq. (B2b). The subscripts \( n \) and \( p \) refer to the carrier type:

\[
U_{nk}^{pp} = U_{nk'k} - U_{npk'}, \\
U_{nk}^{pp} = U_{nk'k} - U_{npk'}.
\]

It remains to find expressions for \( U_{nk'k} - U_{npk'} \) and \( U_{npk'} - U_{nk'k} \).

The recombination rate for holes at the \( k \)th quantum state is the difference in the rate of hole capture and the rate of hole emission. A rate of hole capture from the \( k' \) state to the \( k \) state may be expressed as being proportional to the number of holes at the \( k' \) level and the number of electrons at the \( k \)th level. The constant of proportionality is \( c_p(E_{k'}, E_k) \). Similarly, the rate of hole emission from the \( k \) state to the \( k' \) state is proportional to the number of holes in the \( k \) state and the number of electrons on the \( k' \) state. Again the constant of proportionality is \( e_n(E_k, E_{k'}) \):

\[
\sum_{k'=1, k' \neq k}^{m} U_{nk'k} - U_{npk'} = \sum_{k'=1, k' \neq k}^{m} c_p(E_{k'}, E_k)N_k N_{k'}[1 - f_{k'}(t)] - e_n(E_k, E_{k'})N_k N_{k'}[1 - f_{k'}(t)].
\]

At equilibrium the recombination rate \( U_{nk'k} - U_{npk'} \) is zero and \( E_{Fk} = E_{Fk'} \) = \( E_F \), where \( E_F \) is the equilibrium Fermi level. From Eq. (B3) and the orthogonality relation Eqs. (A4) the following may be established from the given equilibrium condition:

\[
e_n(E_k, E_{k'}) = c_n(E_{k'}, E_k)\exp\left(\frac{E_{k'} - E_k}{kT}\right).
\]

On substitution of Eq. (B4) into Eq. (B3) the recombination rate for holes at the \( k \) state is given as

\[
\sum_{k'=1, k' \neq k}^{m} U_{nk'k} - U_{npk'} = \sum_{k'=1, k' \neq k}^{m} c_p(E_{k'}, E_k)N_k N_{k'} f_{k'}(t)[1 - f_k(t)] - \exp\left(\frac{E_{k'} - E_k}{kT}\right)[1 - f_k(t)]f_{k'}(t).
\]

Referring to Fig. 2, the rate of change of electron occupation at the \( k \) state from the \( k' \) state may be expressed as

\[
\sum_{k'=1, k' \neq k}^{m} U_{nk'k}^{pp} - U_{nk'k}^{pp} = \sum_{k'=1, k' \neq k}^{m} (U_{nk'k} - U_{npk'}) - (U_{npk'} - U_{nk'k}).
\]

Since the average of the \( c_n(E_{k'}, E_k) \) over a band of states cannot be taken, as the \( k' \) level is discrete, let the capture coefficient for the transition of an electron from the \( k' \) state to the \( k \) state be \( C_{nk'k} \). Similarly, the coefficient \( c_n(E_{k'}, E_k) \) is written as \( C_{nk'k} \). Substituting Eqs. (B5) and (B8) into Eq. (B9) yields an expression Eq. (B10) for the carrier transitions between the coupled states:
\[
\sum_{k'=1, k'+k}^{m} U_{k'k}^{\rho\rho} = \sum_{k'=1, k'+k}^{m} N_k N_{k'} \left[ C_{nk'k} + C_{pk'k} \right] \\
\times \exp \left( \frac{E_{k'} - E_k}{kT} \right) f_{k'}(t) \left[ 1 - f_{k}(t) \right] \\
- \left[ C_{nk'k} \exp \left( \frac{E_{k'} - E_k}{kT} \right) + C_{pk'k} \right] f_{k'}(t) \left[ 1 - f_{k}(t) \right],
\]
\] (B10)

where

\[
f_{k}(t) = f_{ok} + \Delta f_{k}(t),
\] (B11a)

\[
f_{k'}(t) = f_{ok'} + \Delta f_{k'}(t),
\] (B11b)

and

\[
\exp \left( \frac{E_{k'} - E_k}{kT} \right) = \frac{f_{ok}}{f_{ok'} (1 - f_{ok})},
\] (B12a)

\[
\exp \left( \frac{E_{k} - E_{k'}}{kT} \right) = \frac{f_{ok}}{f_{ok'} (1 - f_{ok})}.
\] (B12b)

and at equilibrium the state Fermi levels equal the equilibrium Fermi level \(E_F\), with \(E_{Fk} = E_F\) and \(E_{Fk'} = E_F\),

\[
f_{ok} = \frac{1}{1 + \exp \left( \frac{E_{ok} - E_k}{kT} \right)},
\] (B13a)

\[
f_{ok'} = \frac{1}{1 + \exp \left( \frac{E_{ok'} - E_{k'}}{kT} \right)}.
\] (B13b)

or for example, in terms of \(n_{po}\) (p type) and \(n_{1k}\) (n type):

\[
f_{ok} = \frac{n_{po}}{n_{1k} + n_{po}},
\] (B14a)

\[
f_{ok'} = \frac{n_{po}}{n_{1k'} + n_{po}}.
\] (B14b)

Substituting Eqs. (B11) and (B12) into Eq. (B10) yields the component contribution to the rate of localized state change in occupancy from other localized states:

\[
\sum_{k'=1, k'+k}^{m} N_k N_{k'} \left[ C_{nk'k} + C_{pk'k} \right] \Delta f_{k}(t_0) \\
+ \left[ C_{nk'k} \Delta f_{k}(t_0) + C_{pk'k} \right] \Delta f_{k'}(t_0) \\
+ \left[ C_{nk'k} \Delta f_{k}(t_0) + C_{pk'k} \right] \Delta f_{k'}(t_0),
\]
\] (B15)

where

\[
\kappa_{kk'}^{\rho} = \frac{f_{ok'}}{f_{ok}},
\] (B16a)

\[
kappa_{kk'}^{n} = -\frac{1}{f_{ok'}},
\] (B16b)

\[
kappa_{kk'}^{p} = -\frac{1}{f_{ok'}},
\] (B16c)

\[
kappa_{kk'}^{p} = \frac{f_{ok}}{f_{ok'}},
\] (B16d)

and

\[
\mu_{k}^{\rho} = \frac{f_{ok'} - f_{ok}}{f_{ok}(1 - f_{ok})},
\] (B17a)

\[
\mu_{k}^{p} = \frac{f_{ok'} - f_{ok}}{f_{ok}(1 - f_{ok})}.
\] (B17b)

In Eq. (B15), for each \((k', k)\) pair, a term in \(\Delta f_{k}(t_0) \Delta f_{k'}(t_0)\) occurs which is nonzero for the simultaneous intersections of \(\Delta f_{k}(t_0)\) and \(\Delta f_{k'}(t_0)\) with \(\Delta n_{nm}(t_0)\) for some \(t_0 > 0^\circ\). This simultaneous intersection only occurs for high excess carrier concentrations. Hence, for this condition Eq. (B1) together with Eq. (B15) may be expressed as Eq. (B18):

\[
\frac{N_k}{\Delta n(0)} \frac{d\Delta f_{k}(t)}{dt} = \left[ \Delta n_{nm}(t_0) \right] \left[ \gamma_k - \sigma_k \right] \frac{N_k}{\Delta n(0)} \Delta f_{k}(t_0) \\
- \left[ \Delta p_{nm}(t_0) \right] \left[ \alpha_k - \beta_k \right] \Delta n_{nm}(t_0) \Delta f_{k}(t_0) \\
+ \sum_{k'=1, k'+k}^{m} \left[ C_{nk'k} \kappa_{kk'}^{\rho} + C_{pk'k} \kappa_{kk'}^{p} \right] \Delta f_{k}(t_0) \\
+ \sum_{k'=1, k'+k}^{m} N_k \frac{N_k}{\Delta n(0)} \left[ \Delta f_{k}(t_0) \right] \Delta f_{k'}(t_0) \Delta n_{nm}(t_0) \Delta f_{k'}(t_0),
\]
\] (B18)

APPENDIX C: EVALUATION OF \(\eta_{nk}(t_0), \eta_{pk}(t_0)\)

For low injection \(\eta_{nk}(t_0), \eta_{pk}(t_0)\) at \(t_0 = 0^\circ\) are both zero. It remains to find a value for \(\eta_{nk}(t_0), \eta_{pk}(t_0)\) at \(t_0 > 0^\circ\). Following the derivation similar to Ref. 4 the relevant expressions are given below. For example, for the \(k\) state a requirement for the selection of \(\eta_{nk}(t_0) = \Delta f_{k}(t_0) = \Delta n_{nm}(t_i)\) (p type) is found from the charge neutrality relation with \(\Delta p(0) = \Delta n(0)\) as follows:

\[
\Delta p_{nm}(t_0) = \left( 1 + \frac{N_k}{\Delta n(0)} \right) \Delta n_{nm}(t_0),
\] (C1)

where for p type

\[
\phi_{k} = \left( 1 + \frac{N_k}{\Delta n(0)} \right)
\] (C2)

and for n type

\[
\phi_{n} = \left( 1 - \frac{N_k}{\Delta n(0)} \right).
\] (C3)

A relationship between the differentials is as follows:
Having established this relationship for a given \( k \) Eqs. (18a) and (18b) may be solved for \( \eta_k(t_{l_k}) \) given by Eq. (C5) with \( \phi = \phi_k \) for the case of \( r \) type or \( n \) type, to simplify notation and, \( \alpha_{0k}, \beta_{0k}, \gamma_{0k}, \) and \( \sigma_{0k} \) given by Eqs. (8):

\[
\eta_k(t_{l_k}) = \frac{\phi_0 \alpha_{0k} - \beta_{0k}) - (\phi \gamma_{0k} - \phi^2 \sigma_{0k})}{\phi \Delta n(0)(1 - \phi)(C_{nk} + C_{pk})}.
\]

The value of \( \eta_p(t_{l_k}) = \Delta n_{mm}(t_{l_k}) \) is evaluated from Eq. (C1). Solving the following quadratic Eqs. (C6) and (C7) resulting from Eq. (C5) with \( \Delta \eta_k(t_{l_k}) = 0 \), corresponding to the linear (constant) region of variation of \( \omega \) with \( \Delta n(0) \) [see Eq. (23)], indicates the onset of the nonlinear region. For \( \Delta n(0) > 0 \), the onset of the nonlinear region is termed \( \Delta n(0)_{\text{critical}} \). For \( p \) type \( \Delta n(0)_{\text{critical}} \) is evaluated from

\[
0 = \Delta n^2(0)[(\alpha_{0k} - \beta_{0k}) - (\gamma_{0k} - \sigma_{0k})]
\]

\[
+ \Delta n(0)[N_k \alpha_{0k} - N_k \gamma_{0k} + 2N_k \sigma_{0k}] + N_k^2 \sigma_{0k},
\]

and for \( n \) type \( \Delta n(0)_{\text{critical}} \) is evaluated from

\[
0 = -\Delta n^2(0)[(\alpha_{0k} - \beta_{0k}) - (\gamma_{0k} - \sigma_{0k})]
\]

\[
+ \Delta n(0)[N_k \alpha_{0k} - N_k \gamma_{0k} + 2N_k \sigma_{0k}] + N_k^2 \sigma_{0k}.
\]